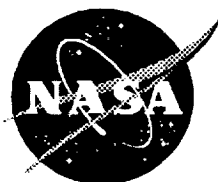


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# A PHASE EQUATION APPROACH TO BOUNDARY LAYER INSTABILITY THEORY: TOLLMIEEN SCHLICHTING WAVES

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## Abstract

Our concern is with the evolution of large amplitude Tollmien-Schlichting waves in boundary layer flows. In fact the disturbances we consider are of a comparable size to the unperturbed state. We shall describe two-dimensional disturbances which are locally periodic in time and space. This is achieved using a phase equation approach of the type discussed by Howard and Kopell (1977) in the context of reaction-diffusion equations. We shall consider both large and  $O(1)$  Reynolds numbers flows though, in order to keep our asymptotics respectable, our finite Reynolds number calculation will be carried out for the asymptotic suction flow. Our large Reynolds number analysis, though carried out for Blasius flow, is valid for any steady two-dimensional boundary layer. In both cases the phase equation approach shows that the wavenumber and frequency will develop shocks or other discontinuities as the disturbance evolves. As a special case we consider the evolution of constant frequency/wavenumber disturbances and show that their modulational instability is controlled by Burgers equation at finite Reynolds number and by a new integro-differential evolution equation at large Reynolds numbers. For the large Reynolds number case the evolution equation points to the development of a spatially localized singularity at a finite time. The three-dimensional generalizations of the evolution equations is also given for the case of weak spanwise modulations.

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# 1 Introduction

Most boundary layers of practical importance are susceptible to a variety of instability mechanisms which lead to the onset of transition to turbulence. Usually more than one mechanism will be operational in any particular case, and a full understanding of how transition occurs will require a detailed understanding of the nonlinear interaction of the different modes of instability. Here we will concern ourselves with the strongly nonlinear evolution of a slowly-varying Tollmien-Schlichting wave system. In the first instance we consider lower branch Tollmien-Schlichting waves which are known to be governed by triple-deck theory e.g. Smith (1979a,b), Hall and Smith (1984), Smith and Burggraf (1985). Then we shall consider the corresponding problem at finite Reynolds numbers. It is worth pointing out that the approach we use here can be used to describe other modes of instability and is in fact based on ideas given some years ago by Howard and Kopell (1977) who were interested in the evolution of nonlinear wave systems in reaction-diffusion equations.

The first application of triple-deck theory to describe the linear and nonlinear growth of lower branch Tollmien-Schlichting waves is apparently due to Smith (1979a,b) though Lin (1966) clearly recognized the appropriate large Reynolds number scalings for Tollmien-Schlichting waves long before triple-deck theory was invented. The investigation of Smith (1979a) showed how nonparallel effects could be taken care of in a self-consistent manner using asymptotic methods. Previously Gaster (1974) used a successive approximation procedure to tackle the same kind of problem. Subsequently Smith (1979b) showed how the nonlinear growth of Tollmien-Schlichting waves could be taken care of using triple-deck theory. However the results of Smith (1979b), and the subsequent extension to three-dimensional modes by Hall and Smith (1984), are confined to the weakly nonlinear stage where an ordinary differential amplitude equation describes the initial stage of the bifurcation from a disturbance free state. Some years later Smith and Burggraf (1985) discussed the high frequency limit of the lower branch triple-deck problem and uncovered a sequence of nonlinear structures governing a sequence of successively more nonlinear wave interactions. Subsequently Smith and Stewart (1987) investigated the interaction of three-dimensional modes at high frequencies and obtained excellent agreement with the experiments of Kachanov and Levechenko (1984), though Khokhlov (1994) in a recent theoretical investigation argues that the work of Smith and Stewart (1987) needs some modification.

In the first instance we shall restrict our attention to two-dimensional waves and determine how the wavenumber and frequency of a wavesystem may be found as it moves through a growing boundary layer. This problem has not yet been addressed. Intuitively one would expect that a small amplitude wave would evolve from its weakly nonlinear form into a larger amplitude state until it is described by the Smith-Burggraf structure at sufficiently large values

of the local frequency of the disturbance. Our calculations show that this is not the case and indeed show that at high frequencies locally periodic forms of the modes found by Smith and Burggraf probably do not exist. Certainly it would appear that they do not connect to the weakly nonlinear state discussed by Smith (1979b).

The asymptotic structure we use is based on the so-called 'phase-equation' approach used so successfully to describe large amplitude Bénard convection in large containers by, amongst others, Kramer, Ben Jacob, Brand and Cross (1982), Cross and Newell (1984), Newell, Passot and Lega (1993). Using this approach it has been possible to describe the experimentally observed slowly varying planform of Benard convection. Thus, for example, the dislocation of convection rolls is now reasonably well understood using the phase equation approach. Interestingly enough it turns out that the essential ideas of this approach had been elucidated in the context of traveling wave instabilities several years earlier, see Howard and Kopell (1977) and indeed the method can be found in Whitham (1974). The evolution of traveling waves in a Blasius boundary layers is the subject of the first part of this work and not surprisingly the analysis to be used has similarities with that of the latter authors.

The essential idea behind the phase equation approach may be explained in the following manner. Suppose there exists some flow which is unstable to a traveling wave disturbance of wavenumber  $\alpha$  and frequency  $\Omega$ . For a fully nonlinear disturbance the frequency  $\Omega$  will be a function of  $\alpha$  which itself can be thought of as a function of  $\Delta$ , a measure of the size of disturbance. If we let  $\Delta$  tend to zero then, for small  $\Delta$ , the quantities  $\alpha$  and  $\Omega$  will differ from their linear neutral values by  $O(\Delta)^2$  so that finite amplitude disturbances begin as supercritical bifurcations from the basic state. For  $O(1)$  values of  $\Delta$  the quantities  $\alpha$  and  $\Omega$  are accessible only by numerical means, see for example Herbert (1977) for details of the computation of  $\alpha$  and  $\Omega$  for Tollmien-Schlichting waves in plane Poiseuille flow or Conlisk, Burggraf and Smith (1987) for a similar calculation for Tollmien-Schlichting waves in Blasius boundary layers at large Reynolds numbers. In some cases the frequency of the waves is zero and, the wavenumber of the disturbances may be sensibly held fixed when the control parameter or disturbance size is varied, see for example Hall (1988) for a discussion of the fully nonlinear Görtler problem in a growing boundary layer. For a traveling wave disturbance in a growing boundary layer we expect that the wavenumber and frequency of the disturbance should change as it propagates into locally less or more unstable parts of the flow. The phase equation approach provides a rational framework for following such an evolution. If the wave has local frequency and wavenumber which are  $O(1)$  with respect to the variables  $t$  and  $x$  we introduce slowly varying variables  $T$  and  $X$  by writing

$$T = \delta t, \quad X = \delta x,$$

and we now think of  $\alpha$  and  $\Omega$  as being functions of  $X$  and  $T$ . Thus we may introduce the



phase function  $\Theta(X, T)$  defined by  $\Theta = \theta(X, T)/\delta$  with

$$\alpha = \theta_X, \Omega = -\theta_T$$

and as a consequence the wave system evolves such that

$$\alpha_T + \Omega_X = 0. \quad (1.1)$$

Partial derivatives with respect to  $x$  and  $t$  must then be replaced using

$$\frac{\partial}{\partial x} \rightarrow \alpha \frac{\partial}{\partial \Theta} + \delta \frac{\partial}{\partial X}, \quad \frac{\partial}{\partial t} \rightarrow -\Omega \frac{\partial}{\partial \Theta} + \delta \frac{\partial}{\partial T},$$

and if we then equate terms of  $O(\delta)^0$  we recover the unmodulated equations of motion with  $\alpha$  and  $\Omega$  playing the role of wavenumber and frequency. Thus the leading order problem using the phase equation approach is simply the unmodulated case with  $\alpha$  and  $\Omega$  being functionally related in order that the system, with a given disturbance size, has a solution. At next order (usually  $O(\delta)$  but in fact  $O(\delta^{1/3})$  in triple deck problems) we obtain a linearized inhomogeneous form of the leading order problem. Due to the invariance of the problem under a translation in the  $x$  direction it is easy to see that the linearized homogeneous form of this system has a non-trivial solution so that some solvability condition must be satisfied if the inhomogeneous problem is to have a solution. This solvability condition is satisfied by introducing an expansion of  $\Omega$  in appropriate powers of  $\delta$ . The solution of this problem then enables us to write down the asymptotic form of (1.1) upto the second order. This procedure can be continued in principle to any order and the coefficients in the expansion of  $\Omega$  are found as solvability conditions at each order. The evolution of a given wave system can then be found by the solution of the calculated asymptotic approximation to (1.1). We find that (1.1) takes on a particularly simple form if the wave system has fixed wavenumber and frequency at leading order. We shall in this paper calculate (1.1) correct upto second order for both wavesystems governed by triple-deck theory and those satisfying the two-dimensional Navier-Stokes equations. We can show that at high Reynolds numbers (1.1) may then be reduced to the form

$$\frac{\partial \Lambda}{\partial \tau} + \Lambda \frac{\partial \Lambda}{\partial \xi} = -\frac{\partial}{\partial \xi} \int_{\xi}^{\infty} \frac{\frac{\partial \Lambda}{\partial s}}{(s - \xi)^{1/3}} ds. \quad (1.2)$$

This is in effect the evolution equation for a wavepacket of large amplitude Tollmien-Schlichting waves in a growing boundary layer.

At finite Reynolds numbers the modulation equation corresponding to (1.2) is

$$\Lambda_{\tau} + \Lambda \Lambda_{\xi} = \pm \Lambda_{\xi \xi}. \quad (1.3)$$

Thus at finite Reynolds numbers Burgers equations controls the slow dynamics of a two-dimensional wavesystem. We will show that (1.2) and (1.3) can be generalized to allow for a

weak spanwise dependence of the modulation and, surprisingly, it turns out that the generalizations in both cases are achieved by adding a term proportional to  $\Lambda_{\zeta\zeta}$  to both equations. Here  $\zeta$  is a slow spanwise variable.

The procedure adopted in the rest of this paper is as follows: In §2 we derive the phase equation for two-dimensional triple-deck problems. In §3 we describe the numerical work required to determine the quantities appearing in the equation. In §4 we look at the special case of almost uniform wavetrains and derive (1.2). In §5 we show how (1.2) can be derived by a more conventional multiple scale approach directly from the triple-deck equations. The phase equation approach for a boundary layer at finite Reynolds numbers is then discussed in §6. The modulation equation (1.3) is derived in that section as a special case for almost uniform wavetrains. Finally in §7 we draw some conclusions and give the generalized form of the evolution equations which account for weak spanwise modulations and nonparallel effects.

## 2 Derivation of the phase equation for 2D triple deck problems

Our concern is with the structure of fully nonlinear solutions of the triple-deck equations governing the evolution of two-dimensional Tollmien-Schlichting waves in incompressible boundary layers. Following the usual notation, e.g. Smith (1979a), the appropriate differential equations in scaled form are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (2.1a)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2}, \quad (2.1b)$$

which must be solved subject to

$$u = v = 0, \quad y = 0, \quad (2.2a)$$

$$u \sim y + A(x, t), \quad y \rightarrow \infty, \quad (2.2b)$$

$$p = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\partial A}{\partial s} \frac{ds}{(x-s)}. \quad (2.2c)$$

The displacement function  $A$  and the pressure  $p$  depend only on  $x$  and  $t$  and if we wish to consider other flows the pressure displacement law (2.2c) must be modified accordingly. Linear Tollmien-Schlichting waves correspond to perturbing  $u$  in the form

$$u = y + U(y)e^{i\alpha\{x - \frac{\Omega t}{\alpha}\}}, \quad (2.3)$$

with  $U$  small and, from Smith (1979a), the eigenrelation takes the form

$$A'_i(\xi_0) = (i\alpha)^{1/3} \alpha \int_{\xi_0}^{\infty} A_i(\eta) d\eta. \quad (2.4)$$

where  $A_i$  is the Airy function and  $\xi_0 = -\frac{i\Omega}{(i\alpha)^{2/3}}$ . Solutions of (2.4) with  $\Omega$  complex and  $\alpha$  real show that neutral stability occurs for  $\Omega \simeq 2.298$ ,  $\alpha \simeq 1.001$ , and that  $\Omega_i$  is positive for all frequencies greater than the neutral value. In the high frequency limit it can be shown from (2.4) that

$$\alpha = \Omega + O(\Omega^{-1/2}) \quad (2.5)$$

The above limit was discussed in detail by Smith and Buggraff (1985) who investigated the possible nonlinear structures which emerge in that limit. The structures found by Smith and Burggraf depend crucially on the fact that the right hand side of (2.5) is complex only at order  $\Omega^{-1/2}$  so that, even though a wave is never neutral, its growth can be balanced at higher order by nonlinear effects. Here our interest is with the case  $\alpha = O(1)$ ,  $\Omega = O(1)$ , but we shall allow for a slow evolution of the wavesystem as it moves through the boundary layer. The essential details of our approach are to be found in Howard and Kopell (1977) who were concerned with slowly varying waves in reaction diffusion systems. As a first step we introduce slow time and space variables,  $T$  and  $X$ , by writing

$$T = \delta t, \quad (2.6a)$$

$$X = \delta x, \quad (2.6b)$$

where  $\delta$  is a small positive parameter. We shall investigate the evolution of a fully nonlinear wavelike solution of (2.1)-(2.2), but allow the wavenumber and frequency to be slow functions of  $X$  and  $T$ .

In order to describe such a structure we introduce a phase function  $\theta(x, t)$  such that the wavenumber and frequency of the wave are defined by

$$\alpha = \frac{\partial \theta}{\partial x}, \quad \Omega = -\frac{\partial \theta}{\partial t}. \quad (2.7)$$

The wavenumber and frequency must therefore satisfy

$$\frac{\partial \alpha}{\partial t} + \frac{\partial \Omega}{\partial x} = 0, \quad (2.8)$$

and (2.8) therefore corresponds to the conservation of phase. Now we shall assume, following Howard and Kopell (1977), that  $\alpha$  and  $\Omega$  are functions only of  $X$  and  $T$ . In that case (2.8) reduces to

$$\frac{\partial \alpha}{\partial T} + \frac{\partial \Omega}{\partial X} = 0. \quad (2.9)$$

and it is then convenient to write the phase variable  $\Theta = \delta^{-1}\theta(X, T)$ . The  $x$  and  $t$  derivatives in (2.1)-(2.2) must then be transformed according to

$$\frac{\partial}{\partial x} \rightarrow \alpha \frac{\partial}{\partial \Theta} + \delta \frac{\partial}{\partial X}, \quad (2.10a)$$

$$\frac{\partial}{\partial t} \rightarrow -\Omega \frac{\partial}{\partial \Theta} + \delta \frac{\partial}{\partial T}. \quad (2.10b)$$

We seek a locally wavelike solution of (2.1)-(2.2) and impose periodicity in the phase variable  $\Theta$ . It remains now for us to find a small  $\delta$  solution of the full triple deck problem (2.1)-(2.2). At first sight, in view of (2.10), we would expect to develop a solution of that system in terms of  $\delta$ . However, it turns out that for  $\delta \ll 1$  the leading order approximation to (2.1)-(2.2) has a mean term correction which depends on the slow variable  $X$ . This mean flow is reduced to zero in an outer  $O(\delta^{-1/3})$  boundary layer. The expansions must therefore proceed in powers of  $\delta^{1/3}$  and we therefore write

$$\Omega = \Omega_0 + \delta^{1/3}\Omega_1 + \dots, \quad (2.11a)$$

$$u = u_0(X, T, \Theta, Y) + \delta^{1/3}u_1(X, T, \Theta, Y) + \dots, \quad (2.11b)$$

$$v = v_0(X, T, \Theta, Y) + \delta^{1/3}v_1(X, T, \Theta, Y) + \dots, \quad (2.11c)$$

$$p = p_0(X, T, \Theta) + \delta^{1/3}p_1(X, T, \Theta) + \dots, \quad (2.11d)$$

$$A = A_0(X, T, \Theta) + \delta^{1/3}A_1(X, T, \Theta) + \dots. \quad (2.11e)$$

The leading order problem is then found to be

$$\alpha \frac{\partial u_0}{\partial \Theta} + \frac{\partial v_0}{\partial y} = 0, \quad (2.12a)$$

$$-\Omega_0 \frac{\partial u_0}{\partial \Theta} + \alpha u_0 \frac{\partial u_0}{\partial \Theta} + v_0 \frac{\partial u_0}{\partial y} = -\alpha \frac{\partial p_0}{\partial \Theta} + \frac{\partial^2 u_0}{\partial y^2}, \quad (2.12b)$$

$$u_0 = v_0 = 0, \quad y = 0, \quad (2.12c)$$

$$u_0 \sim y + A_0(X, \Theta, T) \quad y \rightarrow \infty, \quad (2.12d)$$

$$p_0 = \frac{1}{\alpha\pi} \int_{-\infty}^{\infty} \frac{\frac{\partial A}{\partial \psi} d\psi}{(\Theta - \psi)}. \quad (2.12e)$$

Hence the leading order problem is obtained from the full two-dimensional problem by restricting attention to solutions in the form of traveling waves of local wavenumber  $\alpha$  and frequency  $\Omega_0$ . This specifies a nonlinear eigenvalue problem

$$\Omega_0 = \Omega_0(\alpha), \quad (2.13)$$

which must be determined numerically. At this stage we assume that (2.13) and the corresponding nonlinear eigenfunctions  $u_0, v_0, p_0$  and  $A_0$  are known. We further note  $A_0$  may be written in the form

$$A_0 = \bar{A}_0(X, T) + \tilde{A}_0(\Theta, X, T) \quad (2.14)$$

where  $\tilde{A}_0$  has zero mean with respect to  $\Theta$ . It is necessary to reduce  $\bar{A}_0$  to zero before the main deck is encountered, therefore an outer boundary layer is required. Before we investigate the outer boundary layer it is convenient to discuss the next order system in the  $y = 0(1)$  region. The equations to be satisfied are

$$\alpha \frac{\partial u_1}{\partial \Theta} + \frac{\partial v_1}{\partial y} = 0, \quad (2.15a)$$

$$-\Omega_0 \frac{\partial u_1}{\partial \Theta} + \alpha \left\{ u_0 \frac{\partial u_1}{\partial \Theta} + u_1 \frac{\partial u_0}{\partial \Theta} \right\} + v_0 \frac{\partial u_1}{\partial y} + v_1 \frac{\partial u_0}{\partial y} + \alpha \frac{\partial p_1}{\partial \Theta} - \frac{\partial^2 u_1}{\partial y^2} = \Omega_1 \frac{\partial u_0}{\partial \Theta}, \quad (2.15b)$$

subject to

$$u_1 = v_1 = 0, \quad y = 0, \quad (2.15c)$$

and

$$p_1 = \frac{1}{\alpha\pi} \int_{-\infty}^{\infty} \frac{\frac{\partial A_1}{\partial s}}{(x-s)} ds. \quad (2.15d)$$

In addition we require a condition involving  $u_1$  at the edge of the boundary layer. On the basis of (2.2b) we might expect that,  $u_1 \rightarrow A_1$ ,  $y \rightarrow \infty$ , is the appropriate condition. However  $A_1$  is essentially the displacement function in the main deck and  $u_1$  is modified in the outer boundary layer in which the mean flow correction on the slow scale is reduced to zero. Therefore we must write down the condition

$$u_1 \rightarrow D_1, \quad y \rightarrow \infty, \quad (2.16)$$

where  $D_1$  is to be found in terms of  $A_1$  by a consideration of the outer boundary layer. For large values of  $y$  we have  $u \sim y$  so that the thickness of the outer layer is fixed by the balance

$$y\delta \frac{\partial}{\partial X} \sim \frac{\partial^2}{\partial y^2}.$$

Hence we write

$$\eta = \delta^{1/3} y$$

and now develop an asymptotic solution of (2.1)-(2.2) valid for  $\eta = 0(1)$ . The solution here is similar to that in the main deck for two-dimensional triple-deck problems. We write

$$\begin{aligned} u &= \left\{ \delta^{-1/3} \eta + u_M(X, \eta, T) + \dots \right\} + \left\{ U_0 + \delta^{1/3} U_1 + \dots \right\}, \\ u &= \left\{ \delta^{2/3} v_M(X, \eta, T) + \dots \right\} + \frac{1}{\delta^{1/3}} \left\{ V_0 + \delta^{1/3} V_1 + \dots \right\}. \end{aligned}$$

Here the terms in the first bracket of each expansion correspond to the mean flow. If we substitute the above expansions into (2.1)-(2.2), and make the appropriate changes of variables, then we find that the function  $U_0$  is given by

$$U_0 = \tilde{A}_0 + \delta^{1/3} \tilde{H}_1 + \delta^{1/3} \tilde{A}_0 u_{M\eta} + \dots$$

where  $\tilde{H}_1$  is to be determined. The functions  $u_M$  and  $v_M$  are found to satisfy

$$\begin{aligned} \frac{\partial^2 u_M}{\partial \eta^2} - \eta \frac{\partial u_M}{\partial X} - v_M &= 0, \\ \frac{\partial u_M}{\partial X} + \frac{\partial v_M}{\partial \eta} &= 0, \end{aligned}$$

which are to be solved subject to

$$u_M = \bar{A}_0, v_M = 0, \eta = 0, u_M \rightarrow 0, \eta \rightarrow \infty.$$

The solution of the above problem for the mean flow correction is most easily obtained using a Fourier transform with respect to  $X$ . We find that the function  $u_M$  is such that

$$u_{M\eta}(\eta = 0) = \frac{3^{\frac{1}{3}}}{\Gamma^2\left(\frac{2}{3}\right)} \int_{-\infty}^X \frac{\frac{\partial \bar{A}_0}{\partial s}}{(s - X)^{\frac{1}{3}}} ds. \quad (2.17)$$

If we now match the  $O(\delta^{1/3})$  correction to the wavelike parts of the expansion between the  $y = 0(1)$ ,  $y = 0(\delta^{1/3})$  regions we obtain

$$H_1 + \frac{3^{\frac{1}{3}} \tilde{A}_0}{\Gamma^2\left(\frac{2}{3}\right)} \int_{-\infty}^X \frac{\frac{\partial \bar{A}_0}{\partial s}}{(s - X)^{1/3}} ds = D_1$$

whilst letting  $\eta \rightarrow \infty$  in the definition of  $U_1$  gives

$$H_1 = A_1. \quad (2.18)$$

This closes the problem for the order  $\delta^{1/3}$  problem in the lower layer. Thus in addition to (2.14), (2.15) and (2.16) we have

$$D_1 = A_1 + \frac{3^{\frac{1}{3}} \tilde{A}_0}{\Gamma^2\left(\frac{2}{3}\right)} \int_{-\infty}^X \frac{\frac{\partial \bar{A}_0}{\partial s}}{(s - X)^{1/3}} ds. \quad (2.19)$$

It should also be pointed out that the condition that the mean flow correction  $u_M$  goes to zero at infinity can be relaxed, if required for some  $\bar{A}_0$ , to the condition that  $u_M$  tends to a constant at infinity. This does not alter the condition (2.19). The quantity  $\Omega_1$  is now determined as a solvability condition on the inhomogeneous system specified by (2.15), (2.16) and (2.19). Such a condition is required because of the translational invariance of the leading order wavesystem with respect to  $\Theta$ . Therefore a solution of the homogeneous problem is found by setting  $(u_1, v_1, p_1, A_1) = \frac{\partial}{\partial \Theta}(u_0, v_0, p_0, A_0)$ . In order to find the appropriate solvability condition it is convenient to define  $Z = (p_1, v_1, u_1, u_{1y})^T$ . We then must determine the condition that the inhomogeneous system given below has a solution:

$$\begin{aligned} \frac{\partial Z}{\partial y} &= BZ + C \frac{\partial Z}{\partial \Theta} + \Omega_1 F_1, \\ B &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & u_{0y} & \alpha u_{0\Theta} & v_0 \end{bmatrix} \end{aligned}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & 0 & 0 & 0 \\ \alpha & 0 & \alpha u_0 - \Omega_0 & 0 \end{bmatrix}, F_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ u_{0\Theta} \end{bmatrix},$$

subject to

$$\begin{aligned} u_1 &= v_1 = 0, \quad y = 0, \\ u_1 &\rightarrow A_1 + \frac{3^{\frac{1}{3}} \tilde{A}_0}{\Gamma^2\left(\frac{2}{3}\right)} \int_{\infty}^X \frac{\frac{\partial \tilde{A}_0}{\partial s}}{(s-X)^{1/3}} ds, \quad y \rightarrow \infty, \\ p_1 &= \frac{1}{\alpha\pi} \int_{-\infty}^{\infty} \frac{\frac{\partial A_1}{\partial s}}{X-s_1}. \end{aligned}$$

The system adjoint to the homogeneous form of the above problem is

$$-\frac{\partial \mathbf{J}}{\partial y} = D\mathbf{J} - C^T \frac{\partial \mathbf{J}}{\partial \Theta},$$

with

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_{0y} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & v_0 \end{bmatrix},$$

$$J = (M, N, S, T)^T,$$

and subject to the conditions

$$\begin{aligned} M &= T = 0, \quad y = 0, \\ S &\rightarrow 0, \quad y \rightarrow \infty, \\ M &\rightarrow M_{\infty}(\Theta), T \rightarrow T_{\infty} y^{-2}, y \rightarrow \infty \quad \text{with} \quad T_{\infty} = \frac{-1}{\alpha^2 \pi} \int_{-\infty}^{\infty} \frac{M}{(x-s)} ds. \end{aligned}$$

The condition that the problem for  $(u_1, v_1, p_1, A_1)$  has a solution is then found to be

$$\Omega_1 = K(\alpha) \int_{\infty}^X \frac{\frac{\partial \tilde{A}_0}{\partial \alpha} \frac{\partial \alpha}{\partial s}}{(s-X)^{1/3}} ds, \quad (2.20)$$

with

$$K(\alpha) = \frac{-\frac{3^{\frac{1}{3}}}{\Gamma^2\left(\frac{2}{3}\right)} \int_0^{2\pi} M_{\infty}(\Theta) p_0 d\Theta}{\int_0^{2\pi} \int_0^{\infty} T \frac{\partial u_0}{\partial \Theta} d\Theta dy}. \quad (2.21)$$

At this stage we can write down the phase conservation equation correct to order  $\delta^{1/3}$ . We obtain

$$\frac{\partial \alpha}{\partial T} + \frac{\partial \Omega_0}{\partial \alpha} \frac{\partial \alpha}{\partial X} = -\delta^{1/3} \frac{\partial \Omega_1}{\partial X} + O(\delta^{2/3}), \quad (2.22)$$

and the expansion procedure given above can in principle be continued to any order. We postpone a discussion of the implications of (2.22) until we have described the results of the calculations required to determine  $\alpha, \Omega_0, \Omega_1$ .

### 3 The numerical work

The system (2.12) is periodic in  $\Theta$  so we seek a solution by expanding for example  $v_0$  in the form

$$v_0 = \sum_{-\infty}^{\infty} v_{0m}(y) e^{im\Theta} \quad (3.1)$$

After eliminating  $p_{0m}$  and some linear terms proportional to  $u_{0m}$  from the  $\Theta$  momentum equation, the equation to determine  $v_{0m}$  may be written in the form

$$\frac{d^4 v_{0m}}{dy^4} - im \{ \alpha u_{00} - \Omega \} v_{0m} = R_m, \quad (3.2)$$

where  $R_m$  is a nonlinear function of  $\{u_{0m}\}, \{v_{0m}\}$ . The equation for the mean part of  $u_0$  is then written as

$$\frac{d^2 u_{00}}{dy^2} = \frac{dI}{dy}, \quad (3.3)$$

where  $I$  is a nonlinear function of  $\{u_{0m}\}, \{v_{0m}\}$ . We used central differences to evaluate the derivatives in (3.2), (3.3) and a solution of the resulting nonlinear system was found by iteration after first restricting  $m$  to be less than say  $M$ . In our iterative technique the right hand sides of (3.2) and (3.3) were evaluated at the previous level of iteration and one boundary condition was replaced by

$$v'_0(\infty) = c_0 + id_0$$

where  $c_0$  and  $d_0$  are prescribed real constants. After the iterations converged we then adjusted  $\alpha$  and  $\Omega$  until the previously ignored boundary condition was satisfied. Note here that, because the solution of (2.12) is unique only upto a phase shift, the values of  $\alpha$  and  $\Omega$  obtained by this procedure are functions only of  $\{c_0^2 + d_0^2\}^{1/2}$ .

The grid size and the value of  $M$  were varied until converged results were obtained. In the following discussion the results presented correspond to  $M = 32$ , and 200 grid points in the  $y$  direction with “infinity” at  $y = 10$ . In Figures 1 and 2 we show the dependence of  $\alpha$  and  $\Omega_0$  on the quantity  $\sqrt{c_0^2 + d_0^2}$  which is a measure of the disturbance size. We see that, as predicted by Smith (1979b), finite amplitude motion begins as a supercritical bifurcation from Blasius flow. In Figure 3 we plot  $\Omega_0$  as a function of  $\alpha$ . The calculations could not be continued beyond the point F shown on the Figure ; we will return to this point later. We further notice that  $\Omega_0$  is a multiple valued function of  $\alpha$  for a range of values of  $\alpha$  and that  $\Omega'_0(\alpha)$  becomes infinite when  $\alpha \sim 1.0145$ . In Figure 4 we show the shear stress as a function of  $\Theta$  for a range values of  $\Omega_0$ . The results shown in this picture suggest a reason why Figure 3 cannot be continued beyond the point F. We see that as the point F is approached the shear stress approaches zero at a point. In Figure 5 we show how the contribution to the shear stress from the different modes varies as  $\Omega_0$  varies. We see that the the higher modes grow rapidly as F is approached. This suggests that the shear stress becomes singular as F is approached.



Beyond the point F our calculations failed to converge because the procedure used to solve (3.2)-(3.3) failed to drive the residuals below the tolerance level,  $10^{-12}$ , used throughout our work. A similar result was found by Conlisk et al (1987) who solved (2.12) by an indirect method. In their calculation the Tollmien-Schlichting waves were first forced by a wall motion and then their properties extrapolated as the forcing was reduced to zero. The results of Conlisk et al (1987) have been plotted in Figure 3 and we see that, on the whole, there is good agreement with our work when our program produced converged results. Some of the larger amplitude results by Conlisk et al were obtained by reducing the tolerance level in their iteration procedure. A similar reduction of the tolerance level in our code enables us to continue Figure 3 for slightly larger disturbance amplitudes but we do not plot them for the following reasons. Firstly we found that a reduction of the tolerance level made our results very sensitive to the grid size. Secondly a reduction of the tolerance level at best only enabled us to continue our calculation until  $\alpha$  was reduced to about 1.018. Furthermore the results of Figure 4 suggest to us that the curve of Figure 3 terminates at a point close to F where all the harmonics are excited and a singularity has been encountered. Therefore it does not seem sensible to plot results obtained by reducing the tolerance level further. Further calculations were carried out at large frequencies in order to find finite amplitude solutions of the type predicted by the Smith-Burggraf theory. Despite a careful search of the parameter regime identified by Smith and Burggraf (1985) no solutions could be found, but this does not mean that they do not exist.

The next calculation required concerned the constant  $\Omega_1$  defined by (2.20). In order to calculate  $\Omega_1$  from (2.20) it is necessary to compute the adjoint function  $J = (M, N, S, T)$ . In fact it is easier to solve the problem for  $(u_1, v_1, p_1, A_1)$  directly and find the value of  $\Omega_1$  which enables all the required boundary conditions to be satisfied. It was easier to compute  $\Omega_1$  in this way because the system for  $(u_1, v_1, p_1, A_1)$  can be solved using essentially the same iteration method as used above for the solution of (2.12). In Figure 6 we show the dependence of  $K$  on the wavenumber  $\alpha$ . The fact that  $K$  is singular at  $\alpha = \alpha_c = 1.0145$  is a direct consequence of the fact that  $\Omega'(\alpha_c) = 0$ . In Figure 7 we show the dependence of  $\bar{A}_0$  on  $\alpha$  and we observe that  $\bar{A}_{0\alpha}$  is respectively negative and positive on the lower and upper branches of Figure 3. Here the upper and lower branches correspond to points on Figure 3 which are respectively above or below E. The singularity in  $\bar{A}_{0\alpha}$  is due to the fact that  $\bar{A}_0$  continues to decrease when  $\alpha$  passes through  $\alpha_c$ . The fact that both  $K$  and  $\bar{A}_{0\alpha}$  change sign at  $\alpha_c$  means that viscous effects have essentially the same destabilizing role on the upper and lower branches when uniform wavetrains are considered; see the following section.

Now let us discuss the implications of our calculations for the evolution equation (2.22) which we recall determines the wavenumber  $\alpha$  correct upto order  $\delta^{\frac{1}{3}}$ . The term on the right hand side of (2.22) is due to viscous effects and the results of Figures 6, 7 imply that viscous

effects are destabilizing. The zeroth order approximation to (2.22) yields

$$\frac{\partial \alpha}{\partial T} + \omega_g \frac{\partial \alpha}{\partial X} = 0 \quad (3.4)$$

where  $\omega_g$  is the group velocity. We see from Figure 3 that the group velocity is negative for the upper branch and positive otherwise. This suggests that the upper branch solutions are physically irrelevant since their energy propagates upstream. In fact, the form of Figures 2,3, and the known result about the stability at small amplitude of Tollmien-Schlichting waves, Smith (1979b), suggests that solutions corresponding to the upper and lower branches would be found to be unstable and stable respectively if a Floquet analysis of them were carried out.

Suppose then that we consider the evolution of disturbances corresponding to the lower branch of Figure 3. If at  $T = 0$  we are given

$$\alpha = \bar{\alpha}(X)$$

then for  $T$  positive we have

$$\alpha = \bar{\alpha}(X - \omega_g(\alpha)T)$$

which determines  $\alpha$  implicitly since  $\omega_g$  is a function of  $\alpha$ . It is well-known, eg Whitham (1974), that for positive  $\omega_g'$  the above solution will become multivalued after a finite time if the initial data has a compressive part. This suggests that finite amplitude Tollmien-Schlichting waves will develop discontinuities in wavenumber and frequency as they propagate downstream. When such shocklike structures develop (3.4) is no longer valid, and the viscous term must be brought into play. We expect that the situation then is similar to that for Burgers equation, see Whitham (1974), where viscous effects smooth out shocklike solutions but do not prevent their development. However until a numerical treatment of (3.4) is carried out such remarks should be treated as speculative. Now we shall concentrate on a case where more analytical progress is possible and investigate nearly uniform wavetrains.

## 4 Uniform wavetrains and their stability

Suppose that  $(\alpha_0, \Omega_0(\alpha_0))$  is some point on the curve shown in Figure 3. The corresponding wave with

$$\Theta = \Theta(\alpha_0 X - \Omega_0(\alpha_0)T)$$

corresponds to a constant frequency/wavenumber solution of the full 2D triple deck problem for Tollmien-Schlichting waves. The stability of this system can be readily investigated by use of the phase equation (2.22). We first write  $\alpha = \alpha_0 + \Delta$ , where  $\Delta$  is small, and then (2.22) becomes

$$\begin{aligned} \frac{\partial \Delta}{\partial T} + \Omega'_0(\alpha_0) \frac{\partial \Delta}{\partial X} + \Omega''_0(\alpha_0) \Delta \frac{\partial \Delta}{\partial X} &= -\delta^{1/3} K(\alpha_0) \frac{\partial \bar{A}_0}{\partial \alpha} \frac{\partial}{\partial X} \int_{-\infty}^X \frac{\frac{\partial \Delta}{\partial s}}{(s - X)^{1/3}} ds \\ &+ O(\delta^{1/3} \Delta^2, \Delta^3). \end{aligned}$$

Note that  $K \bar{A}_{0\alpha}$  is negative on both the lower and upper branches respectively of Figure 3.

We can eliminate the term proportional  $\Omega'_0(\alpha_0)$  by an appropriate Galilean transformation. If we then take  $T = 0(\delta^{-1/3})$ ,  $\Delta \sim \delta^{1/3}$  with  $X = 0(1)$  then, in the limit  $\delta \rightarrow 0$ , a suitably rescaled version of the above equation is

$$\frac{\partial \Lambda}{\partial \tau} + \Lambda \frac{\partial \Lambda}{\partial \xi} = -\frac{\partial}{\partial \xi} \int_{-\infty}^{\infty} \frac{\frac{\partial \Lambda}{\partial s}}{(s - \xi)^{1/3}} ds. \quad (4.1)$$

Therefore the longwave instability of a uniform wavetrain of Tollmien-Schlichting waves is governed by the apparently new evolution equation (4.1).

Suppose now that at  $\tau = 0$  there exists a small initial perturbation  $\Lambda = \Lambda_0(\xi)$ . The linearized form of (4.1) shows that  $\Lambda$  evolves according to

$$\Lambda = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Lambda_0^*(k) \exp \left[ ik\xi + \frac{k^2 \tau}{\Gamma(2/3)(ik)^{2/3}} \right] dk,$$

where  $\Lambda_0^*$  is the transform of the initial data and

$$(ik)^{-2/3} = \begin{cases} e^{-\pi i/3} k^{-2/3}, & k > 0, \\ e^{\pi i/3} |k|^{-2/3}, & k < 0. \end{cases}$$

It can be seen that the ultimate form of the disturbance depends on  $\Lambda_0^*(k)$ . More precisely we see that when  $\Lambda_0(\xi)$  is sufficiently concentrated the solution will develop a singularity at a finite time. Thus for example an initial disturbance of Gaussian form will have  $\Lambda_0^* \sim \exp(-k^2)$  and a bounded solution will occur for all time. However if  $\Lambda_0^* \sim \exp|k|^{4/3}$  the solution will become unbounded after a finite time. This is an important result because it says that a constant wavenumber/frequency solution of the two-dimensional triple-deck problem for

Tollmien-Schlichting waves is always unstable to a long wave instability. This is not uncommon in physical problems, eg the Stokes water wave, nonlinear optics. For a discussion of such modulational instabilities the reader is referred to Whitham (1974).

A possible structure for  $\Lambda$  when this singularity is encountered is obtained by writing

$$\Lambda = (\bar{\tau} - \tau)^\lambda \Psi(\phi), \quad \phi = \frac{\xi - \bar{\xi}}{[\bar{\tau} - \tau]^{3/4}}$$

where the singularity occurs when  $\tau \rightarrow \bar{\tau}_-$  and is localized around  $\xi = \bar{\xi}$ . The constant  $\lambda$  will depend on the initial conditions, but we expect that only solutions with  $\lambda \geq 0$  will be possible. The function  $\Psi$  is then found to satisfy

$$-\lambda \Psi + \frac{3\phi}{4} \frac{d\Psi}{d\phi} = -\frac{d}{d\phi} \int_\phi^\infty \frac{\Psi'(\theta)}{(\theta - \phi)^{1/3}} d\theta. \quad (4.2)$$

We seek a continuous solution of (4.2) with  $\Psi = \Psi_0$ , a constant, for  $\phi > 0$  and allow  $\Psi$  to grow algebraically when  $\phi \rightarrow -\infty$ . It is then convenient to write  $\sigma = -\phi$  and let

$$\Psi(\phi) = \Psi(-\sigma) = \Psi_0 \{f(\sigma) + 1\}$$

so that  $f(\sigma)$  satisfies

$$\lambda \{f(\sigma) + 1\} - \frac{3}{4} f'(\sigma) = \frac{d}{d\sigma} \int_{-\infty}^\sigma \frac{f'(t) dt}{(\sigma - t)^{1/3}}, \quad -\infty < \sigma < \infty,$$

and then  $f(\sigma) = 0$ ,  $\sigma \leq 0$ , whilst for  $\sigma$  positive  $f$  is determined by

$$\mu \{f(\sigma) - 1 - \sigma f'(\sigma)\} = \frac{4}{3} \frac{d}{d\sigma} \int_0^\sigma \frac{f'(t) dt}{(\sigma - t)^{1/3}}. \quad (4.3)$$

When  $\mu = 4\lambda/3 = 4/3$  this equation has an exact solution proportional to  $\sigma^{4/3}$ . If this solution is written in terms of the original variables we obtain

$$\Psi = \begin{cases} \Psi_0, & \phi > 0, \\ \Psi_0 + \frac{9\sqrt{3}}{8\pi} \Psi_0 (-\phi)^{4/3}, & \phi < 0, \end{cases} \quad (4.4)$$

For other values of  $\mu$  we can find a solution of (4.3) by taking a Laplace transform. After some calculation we find that  $f(\sigma)$  may then be written in the form

$$f(\sigma) = \frac{\mu}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{p\sigma} Q(p)}{I(p)} dp, \quad (4.5)$$

where

$$I(p) = p^{\mu+1} e^{-\Gamma(2/3)p^{4/3}}, \quad Q(p) = \int_p^\infty q^{\mu-1} e^{-\Gamma(2/3)q^{4/3}} dq.$$

It is then easy to show from (4.5) that when  $\sigma \rightarrow \infty$

$$f(\sigma) \sim \sigma^\mu.$$

Thus in terms of the original variables we obtain

$$\Psi \sim (-\phi)^{\frac{4\lambda}{3}}, \quad \phi \rightarrow -\infty.$$

which incidentally confirms the exact solution (4.4). Thus we have constructed solutions of (4.2) for which  $\frac{d\Psi}{d\phi}$  is singular with  $0 < \lambda < 3/4$  whilst the solution for  $\lambda = 0$  corresponds to a finite jump in  $\Psi$  between  $\phi = \pm\infty$  so that the wavenumber perturbation (and therefore the associated velocity displacement function) has a shock structure.

In the absence of a full numerical investigation of the evolution equation (4.1) we cannot say which of the singular solutions will be excited. Indeed nothing in our discussion so far has ruled out the possibility of finite amplitude equilibrium solutions of (4.1). However, if we multiply (4.1) by  $X$  and integrate from  $\xi = -\infty$  to  $\xi = +\infty$ , and use Plancherets formula to simplify the contribution from the right hand side of (4.1), we obtain

$$\frac{d}{d\tau} \int_{-\infty}^{\infty} X^2(\xi, \tau) d\xi = \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^{\infty} r^{1/3} |X^*(r, \tau)|^2 dr \quad (4.6)$$

where  $X^*$  is the Fourier transform of  $X$ . It follows from (4.6) that equilibrium solutions (constant or in the form of traveling waves) are ruled out.

## Breakdown of the full nonlinear problem

Now let us consider a possible breakdown form for the full nonlinear system (4.1). If the breakdown is governed by the inviscid form of the equation then, following Brotherton-Ratcliffe and Smith (1987), we can, after a suitable shift of origin, write

$$\Lambda = |\tau|^{n-1} Q_0(\eta) + \dots$$

with  $\eta = \frac{\xi}{|\tau|^n}$ . The term on the right hand side of (4.1) is then negligible for  $n < 3/4$  and  $Q_0$  is then given implicitly by

$$\eta = -Q_0 - \epsilon_0 Q_0^3, \quad (4.7)$$

with  $\epsilon_0$  a positive constant. However (4.7) only determines  $Q_0$  as a single-valued function of  $\eta$  when  $n = \frac{L}{L-1}$ ,  $L = 3, 5, 7, \dots$  so that this type of structure is not possible. However we can take  $n = 3/4$  in which case  $Q_0$  satisfies

$$\frac{3}{4} \eta Q_0' + \frac{Q_0}{4} + Q_0 Q_0' = -\frac{\partial}{\partial \eta} \int_{\eta}^{\infty} \frac{\partial Q_0}{\partial s} (s - \eta)^{-1/3} ds. \quad (4.8)$$

The above integral equation must then be solved numerically. We postpone that numerical investigation until we have carried out full numerical simulations of (4.1); such an investigation will be reported on in due course.

## 5 A direct evaluation of the wavenumber modulation equation

We shall now use a multiple scale approach to derive (4.1) directly from the two-dimensional triple deck equations (2.1). Again the major effect of the modulation is to introduce a layer of depth  $\delta^{-1/3}$  sitting on top of the lower deck. We again define slow variables  $X$  and  $T$  by writing

$$X = \delta x, \quad (5.1a)$$

$$\bar{T} = \delta t. \quad (5.1b)$$

It is then convenient to define  $\hat{X}, \hat{T}$  by

$$\hat{X} = X - \omega_g T, \quad (5.2a)$$

$$\hat{T} = \delta^{1/3} T, \quad (5.2b)$$

where  $\omega_g$  is a group velocity to be determined at higher order in our expansion procedure. Suppose then that we seek a solution of the triple-deck equations which is periodic in  $\phi = \alpha x - \Omega t$  where  $\alpha$  and  $\Omega$  are now taken to be constant. In the lower deck we expand  $u$  in the form

$$u = \sum_{n=0}^{\infty} \alpha^n \delta^{n/3} B^n(\hat{X}, \hat{T}) \frac{\partial^n u_0(\phi, y)}{\partial \phi^n} + \delta^{4/3} [u_4(\phi, y, \hat{X}, \hat{T})] + \delta^{5/3} u_5(\phi, y, \hat{X}, \hat{T}) + \dots \quad (5.3)$$

together with similar expansions for  $v$  and  $p$ . We note at this stage that the summation term in (5.3) arises because of the translational invariance of a solution of the two-dimensional triple-deck equations. In addition we note that the first correction to the underlying mean state dependent on  $\hat{X}$  arises at  $O(\delta^{4/3})$  in (5.3). It should also be stressed at this stage that  $u_0$ , the  $O(1)$  term in (5.3), is independent of the slow scales  $\hat{X}$  and  $\hat{T}$  and that  $B(\hat{X}, \hat{T})$  is an amplitude function to be determined. Thus in (5.3) we identify the term  $\delta^{1/3} B \frac{\partial u_0}{\partial \phi}$  as a small amplitude perturbation to the periodic flow  $u_0(\phi, y)$ . The eigenfunction  $\frac{\partial u_0}{\partial \phi}$  occurs because of the translational invariance of any  $\phi$ -periodic solution of the triple deck problem. For our purposes here it is sufficient for us to consider the partial differential equations to determine  $(u_4, v_4, p_4)$  and  $(u_5, v_5, p_5)$ . If the expansions for  $u, v, p$  are substituted into the triple deck equations, and the appropriate change of variables made, then we find that  $(u_4, v_4, p_4)$  satisfies

$$\alpha \frac{\partial u_4}{\partial \phi} + \frac{\partial v_4}{\partial y} = -B_{\hat{X}} U_{0\phi}, \quad (5.4a)$$

$$\frac{\partial^2 u_4}{\partial y^2} - \alpha \frac{\partial p_4}{\partial \phi} - \alpha u_0 \frac{\partial u_4}{\partial \phi} - \alpha u_4 \frac{\partial u_0}{\partial \phi} - v_4 \frac{\partial u_0}{\partial y} - v_0 \frac{\partial u_4}{\partial y} = [-\omega_g u_0 + \alpha u_0 u_{0\phi}] B_{\hat{X}}. \quad (5.4b)$$

These equations must be solved subject to  $u_4 = v_4 = 0$ ,  $y = 0$  whilst for large  $y$  the appropriate conditions are

$$u_4 \rightarrow \hat{A}(\phi, \hat{X}, \hat{T}), \quad (5.5a)$$

with

$$p_4 = \frac{1}{\alpha\pi} \int_{-\infty}^{\infty} \frac{\frac{\partial \hat{A}}{\partial s}}{(\phi - s)} ds \quad (5.6a)$$

Since the homogeneous form of the system for  $(u_4, v_4, p_4)$  has the solution  $(u_4, v_4, p_4) = \frac{\partial}{\partial \phi}(u_0, v_0, p_0)$  it follows that a solution exists only if an orthogonality condition is satisfied. This may be written down following the procedure used in §4, it is sufficient here to note that the condition determines the group velocity  $\omega_g$  and that the expansion obtained is identical to that which it was derived in §3 after making a perturbation in the wavenumber. The solution of (5.4) is clearly of the form

$$(u_4, v_4, p_4) = B_{\hat{X}}(U_4, V_4, P_4), \hat{A} = B_{\hat{X}} \tilde{A},$$

where  $(U_4, V_4, P_4)$  and  $\tilde{A}$  are independent of  $\hat{X}, \hat{T}$ . The other main feature to appreciate about the solution of the  $O(\delta^{4/3})$  problem is that at the edge of the  $y = 0(1)$  region  $U_4$  may be expressed in the form

$$U_4 \sim \tilde{A} = (A_M + A_F(\phi))B_{\hat{X}}$$

where  $A_M$  is independent of  $\phi$  and therefore corresponds to a mean flow correction. The reduction to zero of  $A_M$  is achieved in the outer  $O(\delta^{-1/3})$  region in a similar manner to that found in §3. We define the variable  $\eta$  by

$$\eta = \delta^{1/3}y$$

and in the outer  $\delta^{-1/3}$  layer  $u$  is expanded in the form

$$u = \frac{\eta}{\delta^{1/3}} + \sum_{n=1}^{\infty} \alpha^n \delta^{n/3} B^n(\hat{X}, \hat{T}) \frac{\partial^n u_0}{\partial \phi^n}(\phi, \infty) + \delta^{4/3} U_M(\hat{X}, \eta) + \delta^{5/3} U_5 + \dots$$

where  $U_M$  is the mean flow correction driven by  $A_M$ . The mean flow correction in the  $\eta$  direction is  $\delta^2 V_M$  and the linear problem to determine  $(U_M, V_M)$  is found to be

$$\begin{aligned} \frac{\partial U_M}{\partial \hat{X}} + \frac{\partial V_M}{\partial \eta} &= 0, \\ \frac{\partial^2 U_M}{\partial \eta^2} - \eta \frac{\partial U_M}{\partial \hat{X}} - V_M &= 0, \\ U_M &= A_M B_{\hat{X}}, V_M = 0, \eta = 0 \\ U_M &\rightarrow 0, \eta \rightarrow \infty. \end{aligned} \quad (5.7)$$

The system (5.7) is identical to that obtained in Section 2 and therefore may be solved again using a Fourier Transform technique, the solution is not repeated here. The mean flow at order

$\delta^{4/3}$  then interacts with the  $O(\delta^0)$  flow to produce an  $O(\delta^{5/3})$  correction to the outer boundary condition for the disturbed flow in the  $y = 0(1)$  layer. Again the analysis follows closely that of §2 so we do not repeat it here, we find that  $(u_5, v_5, p_5)$  must satisfy

$$\frac{\partial u_5}{\partial \phi} \rightarrow A_5 + \bar{\lambda} \frac{\partial}{\partial \hat{X}} \int_{\hat{X}}^{\infty} \frac{\frac{\partial B}{\partial s}(s, \hat{T})}{(s - \hat{x})^{1/3}} ds,$$

where  $\bar{\lambda}$  is a constant. In the  $y = 0(1)$  region  $(u_5, v_5, p_5)$  is found to satisfy (5.4a,b) but with the right hand sides of these equations replaced by  $[\cdot]$ , and

$$[\cdot] + B_{\hat{T}} u_{0\phi} + B B_{\hat{X}} \left\{ v_4 \frac{\partial u_0}{\partial \phi} + u_0 \frac{\partial u_4}{\partial \phi} + v_4 \frac{\partial u_0}{\partial y} + v_0 \frac{\partial u_4}{\partial y} + \dots \right\}$$

respectively. Here  $[\cdot]$  denotes terms which don't contribute to the solvability condition which the system for  $(u_5, v_5, p_5)$  must satisfy. The required condition is

$$\frac{\partial B}{\partial \hat{T}} + g_1 B \frac{\partial B}{\partial \hat{X}} = \lambda \frac{\partial}{\partial \hat{X}} \int_{\hat{X}}^{\infty} \frac{\frac{\partial B}{\partial S}(S, \hat{T})}{(S - \hat{X})^{1/3}} dS, \quad (5.8)$$

where  $g_1, \hat{\lambda}$  are constants and a suitable change of variables enables us to recover (4.1). Thus  $\Gamma$  in (4.1) can be interpreted as being either a wavenumber or amplitude perturbation and the two approaches are shown to be consistent.

## 6 The phase equation approach at finite Reynolds numbers

In §2 we derived the phase equation for Tollmien-Schlichting waves in a Blasius boundary layer. Such a boundary layer exists only at asymptotically large values of the Reynolds number and it was therefore appropriate to utilize the largeness of the Reynolds numbers in the description of the instability wave. However linear descriptions of the evolution of Tollmien-Schlichting waves at finite Reynolds numbers have been given by for example Gaster (1974). Though such approaches do not give formal asymptotic approximations to the equations of motion it appears that they correctly predict the essential physics of the linear growth of Tollmien-Schlichting waves. Similarly large scale numerical simulations of nonlinear growth of Tollmien-Schlichting waves at finite Reynolds numbers have proved equally successful at reproducing experimental results, e.g Wray and Hussaini (1984). Here we wish to investigate the phase equation approach at finite Reynolds numbers but, in order to keep our asymptotic analysis formally correct, we choose to work with a parallel boundary layer which is an exact solution of the Navier-Stokes equations at all Reynolds numbers. We refer to the asymptotic suction boundary layer which has been investigated in the weakly nonlinear regime by Hocking (1975). Suppose then that the freestream speed is  $U_0$  and the suction velocity is  $V_0$ . We define a reference length  $L = \frac{V}{V_0}$  and define the Reynolds number

$$R = \frac{U_0}{V_0}$$



but we assume that  $R = 0(1)$  in this section. Since we restrict ourselves to two-dimensional disturbances it is convenient for us to define a stream function  $\psi$  and work with the vorticity equation in the form

$$\frac{\partial \nabla^2 \psi}{\partial t} + \frac{\partial(\nabla^2 \psi, \psi)}{\partial(x, y)} = \frac{1}{R} \nabla^4 \psi, \quad (6.1)$$

which is to be solved subject to

$$\psi_y = 0, \quad \psi_x = 1, \quad y = 0, \quad (6.2a)$$

$$\psi_y \rightarrow 1, \quad y \rightarrow \infty \quad (6.2b)$$

In the absence of an instability the stream function  $\psi$  is given by

$$\psi = \psi_0(y) = y + e^{-y} - x. \quad (6.3)$$

This flow is unstable to two-dimensional Tollmien-Schlichting waves for  $R > 54370.0$  and the band of unstable wavenumbers tends to zero when  $R \rightarrow \infty$ . Here we assume  $R$  is  $0(1)$  and assume that an  $0(1)$  amplitude wavesystem is superimposed on (6.3). At finite  $R$  the mean flow driven by the wave system is confined to the boundary layer. Therefore no outer adjustment layer is required even when the wavesystem evolves slowly in the downstream direction, see Hocking (1975) for a discussion of this point. Suppose that  $X, T$  are defined by

$$X = \delta x, \quad T = \delta t,$$

then we define

$$\Theta = \frac{1}{\delta} \theta(X, T), \quad \alpha = \theta_X, \quad \Omega = -\Theta_T,$$

and expand  $\Omega$  in the form

$$\Omega = \Omega_0 + \delta \Omega_1 + \dots \quad (6.4)$$

We seek a solution of (6.1) by writing

$$\psi = \psi_0(X, y, \Theta, T) + \delta \psi_1(X, y, \Theta, T) + \dots$$

and the leading problem is determined by solving

$$\begin{aligned} -\Omega_0 \frac{\partial \nabla_1^2 \psi_0}{\partial \Theta} + \alpha \frac{\partial(\nabla_1^2 \psi_0, \psi_0)}{\partial(\Theta, y)} &= \frac{1}{R} \nabla_1^4 \psi_0, \\ \psi_{0y} &= 0, \quad \alpha \psi_{0\Theta} = 1, \quad y = 0, \\ \psi_{0y} &\rightarrow 1, \quad y \rightarrow \infty, \end{aligned} \quad (6.5)$$

with

$$\nabla_1^2 \equiv \frac{\partial^2}{\partial y^2} + \alpha^2 \frac{\partial^2}{\partial \Theta^2},$$

and periodicity in  $\Theta$ . The required frequency and wavenumber,  $\Omega_0$  and  $\alpha$ , must be found numerically once some measure of the disturbance size is specified. We will not attempt such a calculation here but we note from the finite Reynolds number calculation of Hocking (1975) and the large Reynolds number theory of Smith (1979a,b) that both sub and supercritical bifurcations to finite amplitude Tollmien-Schlichting waves are possible. For the purposes of our discussion we simply assume that the nonlinear eigenrelation  $\Omega_0 = \Omega_0(\alpha, R)$  is known. At order  $\delta$  we find that  $\psi_1$  is determined by

$$-\Omega_0 \frac{\partial \nabla_1^2 \psi_1}{\partial \Theta} - \alpha \frac{\partial(\psi_0, \nabla_1^2 \psi_1)}{\partial(\Theta, y)} - \alpha \frac{\partial(\psi_1, \nabla_1^2 \psi_0)}{\partial(\Theta, y)} - \frac{1}{R} \nabla_1^4 \psi_1 = \Omega_1 \frac{\partial \nabla_1^2 \psi_0}{\partial \Theta} + \frac{\partial \alpha}{\partial X} M(\Theta, X, y). \quad (6.6)$$

where

$$M = \frac{\partial(\psi_0, \nabla_1^2 \psi_0)}{\partial(\alpha, y)} + \alpha^2 \frac{\partial(\psi_0, 2 \frac{\partial^2 \psi_0}{\partial \alpha \partial \Theta})}{\partial(\Theta, y)} + \frac{4\alpha}{R} \frac{\partial^2}{\partial \alpha \partial \Theta} \nabla_1^2 \psi_0.$$

The system must be solved subject to periodicity in  $\Theta$  whilst the boundary conditions in  $y$  are

$$\psi_1 = \frac{\partial \psi_1}{\partial y} = 0, \quad y = 0 \quad (6.7a)$$

$$\psi_1 \rightarrow q(X), \quad y \rightarrow \infty. \quad (6.7b)$$

Here  $q(X)$  represents a mean flow normal to the wall at infinity. This flow is essentially driven through the equation of continuity by the  $O(1)$  streamwise velocity component. A solvability condition is required if (6.6)-(6.7) is to have a solution since the translational invariance of (6.5) means that  $\psi_1 = \frac{\partial \psi_0}{\partial \Theta}$  is a solution of the homogeneous form of (6.6)-(6.7). It is worth pointing out at this stage that, if we were performing a calculation in a region of finite depth, then a pressure eigenfunction would have to be allowed for at leading order in order that  $q$  should be reduced to zero. The requirement for such an eigenfunction is well-known in weakly nonlinear stability theory; see for example Davey, Hocking and Stewartson (1974) or DiPrima and Stuart (1975). The solvability condition can be found by writing

$$\underline{Z} = (\psi, \alpha \psi_\Theta, \psi_y, \nabla_1^2 \psi, \alpha \nabla_1^2 \psi_\Theta, \nabla_1^2 \psi_y)^T,$$

in which case the homogeneous form of (6.6) is

$$\alpha \frac{\partial}{\partial \Theta} A \underline{Z} + \frac{\partial}{\partial y} B \underline{Z} + C \underline{Z} = 0,$$

where  $A, B$  and  $C$  are  $6 \times 6$  matrices defined by

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -R\nabla_1^2\psi_{0y} & \alpha R\nabla_1^2\psi_{0\Theta} & 0 & R[\alpha\psi_{0y} - \Omega] & -R\alpha\psi_{0\Theta} \end{bmatrix}$$

The system adjoint to that given above is

$$-\alpha \frac{\partial}{\partial \Theta} A^T \mathbf{Q} - \frac{\partial}{\partial y} B^T \mathbf{Q} + C^T \mathbf{Q} = 0, \quad (6.8)$$

together with conditions of periodicity in  $\Theta$  and

$$q_5 = q_6, \quad y = 0, \infty. \quad (6.9)$$

The condition that (6.6)-(6.7) has a solution then becomes

$$\Omega_1 = -\frac{\partial \alpha}{\partial X} f_1, \quad (6.10)$$

with

$$f_1 = \frac{\int_0^{\frac{2\pi}{\alpha}} \int_0^\infty M(\Theta, X, y) q_6 d\Theta dy}{\int_0^{\frac{2\pi}{\alpha}} \int_0^\infty \frac{\partial \nabla_1^2 \psi_0}{\partial \Theta} q_6 d\Theta dy}.$$

The phase condition  $\Omega_X + \alpha_T = 0$  correct upto  $O(\delta)$  becomes

$$\alpha_T + \alpha_X \Omega_0(\alpha) = \frac{\partial}{\partial X} \{ \alpha_X \cdot f_1(\alpha) \} \delta. \quad (6.11)$$

In order to examine the stability of a uniform wavestream solution we write

$$\alpha = \alpha_0 + \Delta$$

with  $\Delta \ll \alpha_0$ , and (6.11) after a suitable change of scales becomes

$$\Lambda_\tau + \Lambda \Lambda_\xi = \pm \Lambda_{\xi\xi}. \quad (6.12)$$

Thus we obtain the surprising result that the modulational instability of a two-dimensional wavesystem in a boundary layer at finite Reynolds numbers is governed by Burgers equation.

In (6.12) the  $\pm$  signs correspond to the cases when diffusion effects are stabilizing/destabilizing respectively. Without calculating  $\Omega_1$  we do not know which sign is appropriate for the problem under consideration here so we will discuss both possibilities. However we can simply quote the known results about Burgers equation for each case and the implications for the stability of a uniform wavetrain are essentially the same. A full discussion of results quoted below can be found in Whitham (1974) and Howard and Kopell (1977).

If the positive sign is taken the solution remains bounded for all time and indeed localized or periodic solution of (6.12) tend to zero when  $T$  increases. However even in the diffusively stable case weak shock solutions of (6.12) can develop; a discussion of this possibility is given by Whitham (1974), Howard and Kopell (1977). Thus for any given uniform wavetrain whose instability is governed by (6.12) with the positive sign, an initial disturbance can be found which does not decay to zero at large times. The uniform wavetrain is therefore modulationally unstable.

If the negative sign is taken in (6.12) viscous effects are destabilizing and in fact finite time singularities are developed from a broad range of initial conditions. Again it follows that the uniform wavetrain is unstable.

We conclude that at finite Reynolds numbers modulational effects will either cause a finite time singularity to develop if viscous effects are destabilizing, or cause shock discontinuities in wavenumber and frequency in the stable case. In either case uniform wavetrains of two-dimensional Tollmien-Schlichting waves at finite amplitude should not be observed according to our theory.

## 7 Conclusion

We have used a phase equation approach to determine the evolution of Tollmien-Schlichting waves at large and finite Reynolds numbers. In the high Reynolds number case we found that finite amplitude disturbances, periodic in time and space, apparently exist only for a small range of values of the wavenumber  $\alpha$ . The upper branch in Figure 3 describes modes with negative group velocity and so they are therefore of no physical interest. The lower branch on the other hand corresponds to waves with positive group velocity and for these modes the rate of change of the group velocity with  $\alpha$  is positive. This means that (2.22), the leading order approximation to the phase equation, will develop discontinuities after a finite time for many initial disturbances. It might be anticipated that viscous effects, which appear on the right hand side of (2.22), might smooth out such discontinuities. We cannot be sure if that is the case until a numerical investigation of (2.22) is carried out. However, in §4 we investigated the particular case of uniform wavetrains and found that there viscous effects were destabilizing and it seems likely that this is also the case for (2.22). In §4 we saw that the linearized form

of (4.1) leads to a finite time singularity of the type found by Brotherton-Ratcliffe and Smith (1987). For the full nonlinear problem (4.6) shows that equilibrium solutions are not possible and that an initial perturbation cannot decay to zero at large times. In effect this means that periodic solutions of the triple-deck equations are modulationally unstable. A possible form for the structure of a singular solution of the breakdown of the nonlinear form of (4.1) was found. The structure found was again based on the structure found by Brotherton-Ratcliffe and Smith (1987). The validity of the structure must remain open until a full numerical investigation of (4.1) has been carried out. The question of how the Navier-Stokes equations alter their large Reynolds number structure in order to remove the singularities of (4.1) also remains open.

At finite Reynolds numbers we found that the evolution equation for a periodic wavetrain satisfies Burgers equation. Without extensive calculations we cannot say whether the viscous term in (6.12) has positive or negative sign. If it turns out to be negative then viscous effects are again destabilizing and finite time singularities will occur. If the sign is positive then viscous effects are stabilizing. However (6.12) can be solved exactly by the Cole-Hopf transformation and it is known, Whitham (1974), that even in the stable case shocks will in general develop. Thus we conclude that at large or finite Reynolds numbers a uniform wavetrain of Tollmien-Schlichting waves will break down with a singularity or shock developing after a finite time. This casts some doubt on the validity of large-scale simulations of Tollmien-Schlichting waves using Fourier series expansions in the streamwise direction.

In view of the fact that our analysis has been restricted to the two-dimensional case it is possible that three-dimensional effects might prevent the above predictions from occurring in practice. Nevertheless, experimental observations where the Tollmien-Schlichting wave is driven by a wavemaker suggest that the first step in the transition process is the linear growth of two-dimensional Tollmien-Schlichting waves followed by nonlinear saturation and three-dimensional effects coming into play, see for example Klebanoff et al (1962).

As a first step towards looking at the effect of three-dimensionality let us determine how the evolution equations (4.1), (6.12) are modified if a weak spanwise dependence of the wave is allowed for. We shall determine the modifications using the multiple-scale approach of §5 rather than modify the phase equation formulation. Since §5 concerned Tollmien-Schlichting waves at large Reynolds number let us give the essential details of how the corresponding expansion procedure is formulated at finite Reynolds numbers. In the absence of spanwise modulation the streamwise velocity component is written in the form

$$u = \sum_{n=0}^{\infty} \alpha^n \delta^n \Lambda^n(\hat{X}, \hat{T}) \frac{\partial^n u_0(\phi, y)}{\partial \phi^n} + \delta^2 u_2(\phi, y, \hat{X}, \hat{T}) + \delta^3 u_3(\phi, y, \hat{X}, \hat{T}) + \dots \quad (7.1)$$

Here  $\hat{X} = \delta(x - \omega_g t)$  is a slow variable obtained by moving to a coordinate system moving downstream with the group velocity and  $\phi = \alpha x - \Omega_0 t$ . The summation term arises because of

the translational invariance of a periodic solution in the streamwise direction. At order  $\delta^2$  the group velocity is determined as a solvability condition and  $u_2$  will be proportional to  $\Lambda_{\hat{X}}$ . At order  $\delta^3$  terms proportional to  $\Lambda\Lambda_{\hat{X}}$  and a slow time derivative of  $\Lambda$  will be generated. Viscous effects also come into play to generate terms proportional to  $\Lambda_{\hat{X}\hat{X}}$  and the required solvability condition leads us to (6.12). In the presence of spanwise modulations the procedure described above is modified in the following way. Firstly we define a slow variable

$$\zeta = \delta z$$

where  $z$  is the spanwise variable and allow the amplitude function  $\Lambda$  to depend on  $\zeta$ . The above expansion for  $u$  remains unchanged but the  $O(\delta)$  term in the corresponding pressure expansion drives a spanwise flow of order  $\delta^2$  proportional to  $\Lambda_{\zeta}$ . This spanwise velocity component then generates a term proportional to  $\Lambda_{\zeta\zeta}$  in the  $O(\delta^3)$  continuity equation. Similar terms are generated by spanwise diffusion in the  $x, y$  momentum equations. The solvability condition then is modified to give

$$\Lambda_{\tau} + \Lambda\Lambda_{\xi} = \pm\Lambda_{\xi\xi} + \lambda_1\Lambda_{\zeta\zeta}. \quad (7.2)$$

Here  $\lambda_1$  is a constant whose sign cannot be calculated without solving the  $O(1)$  problem numerically. A similar analysis for the triple-deck case gives

$$\frac{\partial\Lambda}{\partial\tau} + \Lambda\frac{\partial\Lambda}{\partial\xi} = -\frac{\partial}{\partial\xi} \int_{\xi}^{\infty} \frac{\frac{\partial\Lambda}{\partial s}}{(s-\xi)^{1/3}} ds + \lambda_2\Lambda_{\zeta\zeta}. \quad (7.3)$$

with  $\lambda_2$  a constant as the required generalization of (4.1). As a starting point to study the role of three-dimensional effects in the two-dimensional breakdown structure we plan to carry out numerical simulations of (7.2), (7.3).

Finally we close with a brief outline of how (2.22), (4.1) are modified by nonparallel flow effects. Within the framework of (2.1) nonparallel effects manifest themselves through the pressure displacement law which allows for elliptic effects in the streamwise direction. However the slow spatial evolution of the unperturbed shear flow does not enter (2.1) since it can be scaled out of the problem. For the weakly nonlinear growth of Tollmien-Schlichting waves Hall and Smith (1984) showed that nonparallel effects are important when the disturbance amplitude is  $O(R^{-7/32})$  and lead to an amplitude equation of the form

$$\frac{dC}{dX} = XC - (1 + i\bar{\delta})C|C|^2, \quad (7.4)$$

with  $\bar{\delta}$  a real constant. Thus nonparallel effects lead to the term  $XC$  in (7.4) thereby causing the increased linear exponential growth of a small disturbance as it evolves in  $X$ . Any small disturbance amplifying by this means eventually becomes nonlinear and for large  $X$  has  $|C|^2 \sim X$ . Let us now show how related terms can be incorporated into (2.22) and (4.1).

We note that (2.11a) proceeds in powers of  $\delta^{\frac{1}{3}}$  and that the Reynolds number has been effectively scaled out of the problem by our assumption that the Tollmien-Schlichting wavesystem is described by the triple-deck system (2.1). This assumption means that the analysis given so far in this paper is formally valid for  $\delta$  large compared to any positive power of  $\epsilon = R^{-\frac{1}{8}}$ . In order to reveal the effect of boundary layer growth we now relax this condition and see which new terms now play a role in (2.11a). Clearly (2.11a) must include terms proportional to powers of  $\epsilon$  because of the higher order terms in the triple deck expansion but in addition there will be a term proportional to  $\epsilon^3 \delta^{-1} X$  obtained by expanding the streamwise dependence of the unperturbed flow in a Taylor series in the streamwise direction. We therefore now expand the frequency in the form

$$\Omega = \Omega_0 + \delta^{1/3} \Omega_1 + \epsilon \Omega_2 + \epsilon^2 \Omega_3 + \epsilon^3 \delta^{-1} X \Omega_4 + O(\epsilon^3, \epsilon^6 \delta^{-2}, \delta^{\frac{2}{3}}). \quad (7.5)$$

The ordering of the terms in the above expansion depends on the relationship between  $\delta$  and  $\epsilon$ . The term proportional to  $\Omega_4$  is the first one dependent on the nonparallel nature of the basic flow. The first significant distinguished limit arises when  $\delta \sim \epsilon$  when the terms proportional to  $\Omega_3, \Omega_4$  become comparable but still small compared to the  $O(\delta^{\frac{2}{3}})$  term. The next significant stage is when  $\delta$  decreases to  $\epsilon^{\frac{2}{5}}$  in which case the nonparallel term and the  $O(\delta^{\frac{2}{3}})$  term are comparable. However the crucial stage arises when  $\delta$  decreases to  $\epsilon^{\frac{3}{4}}$  in which case the terms proportional to  $\Omega_1, \Omega_4$  play an equal role and viscous and nonparallel effects are comparable. Thus if we write

$$\epsilon^3 = h_1 \delta^{\frac{4}{3}},$$

with  $h_1$  an  $O(1)$  constant then (2.2), correct to second order, now gives

$$\frac{\partial \alpha}{\partial T} + \frac{\partial \Omega_0}{\partial \alpha} \frac{\partial \alpha}{\partial X} = -\delta^{1/3} \left( \frac{\partial \Omega_1}{\partial X} + h_1 \Omega_4 + h_1 X \frac{\partial \Omega_4}{\partial X} \dots \right) \quad (7.6)$$

A further decrease in the size of  $\delta$  means that the term proportional to  $\Omega_1$  should be dropped and nonparallel effects dominate the right hand side of the equation for  $\alpha$ . This completes our description of how the phase equation changes its structure when  $\delta$  is decreased. Note here that when we decrease  $\delta$  we are in effect moving further away from the initial location  $x^* = \bar{x}^*$  so the above different orderings of the right hand sides of (7.5) correspond to moving further downstream. Thus the final form has the right hand side of the phase equation dominated by nonparallel effects.

A similar procedure can be used to determine the appropriate modifications to (4.1) which is the evolution equation for the perturbed wavenumber of an initially uniform wavetrain. Here the crucial scaling brings in nonparallel effects at the same stage as the integral term driven by viscous terms. The appropriate scaling now has

$$\epsilon^3 \delta^{-\frac{5}{3}} = h_2$$

with  $h_2$  an  $O(1)$  constant. Equation (4.1) then becomes

$$\frac{\partial \Lambda}{\partial \tau} + (\Lambda + h_3 \tau) \frac{\partial \Lambda}{\partial \xi} = - \frac{\partial}{\partial \xi} \int_{\xi}^{\infty} \frac{\frac{\partial \Lambda}{\partial s}}{(s - \xi)^{1/3}} ds + h_4. \quad (7.7)$$

Here  $h_3, h_4$  are constants proportional to  $h_2$ , and  $h_4$  can be set equal to zero by a change of dependent variable. Therefore the nonparallel modulational equation for Tollmien-Schlichting waves in a growing boundary layer is (7.6) with  $h_4 = 0$ , the numerical solution of that equation will be reported on in due course.

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## FIGURES

Figure 1 The dependence of  $\alpha$  on  $\sqrt{c_0^2 + d_0^2}$ .

Figure 2 The dependence of  $\Omega_0$  on  $\sqrt{c_0^2 + d_0^2}$ .

Figure 3 The dependence of  $\Omega_0$  on  $\alpha$ . The symbols denote the results of Conlisk et al.

Figure 4 The shear stress as a function of  $\Theta$  for  $\Omega_0 = 2.2995, 2.3041, 2.3125, 2.3245, 2.3398, 2.3575, 2.3763, 2.395, 2.4124, 2.4275$

Figure 5 The shear stress  $u'_{0m}(0)$  as a function of  $\Omega_0$  for the first sixteen modes.

Figure 6 The dependence of  $K$  on  $\alpha$ .

Figure 7 The dependence of  $\bar{A}_0$  on  $\alpha$ .

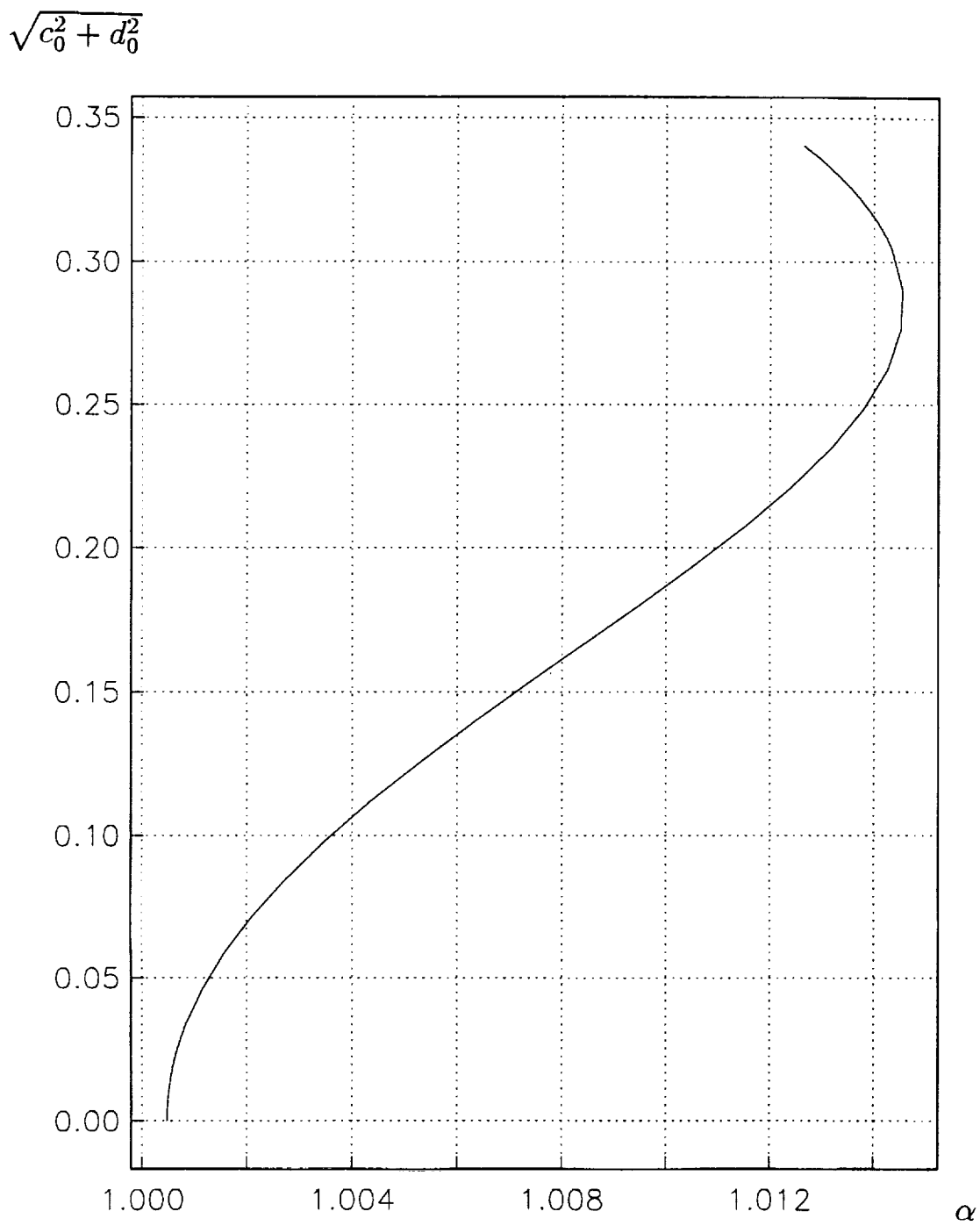


Figure 1 The dependence of  $\alpha$  on  $\sqrt{c_0^2 + d_0^2}$ .

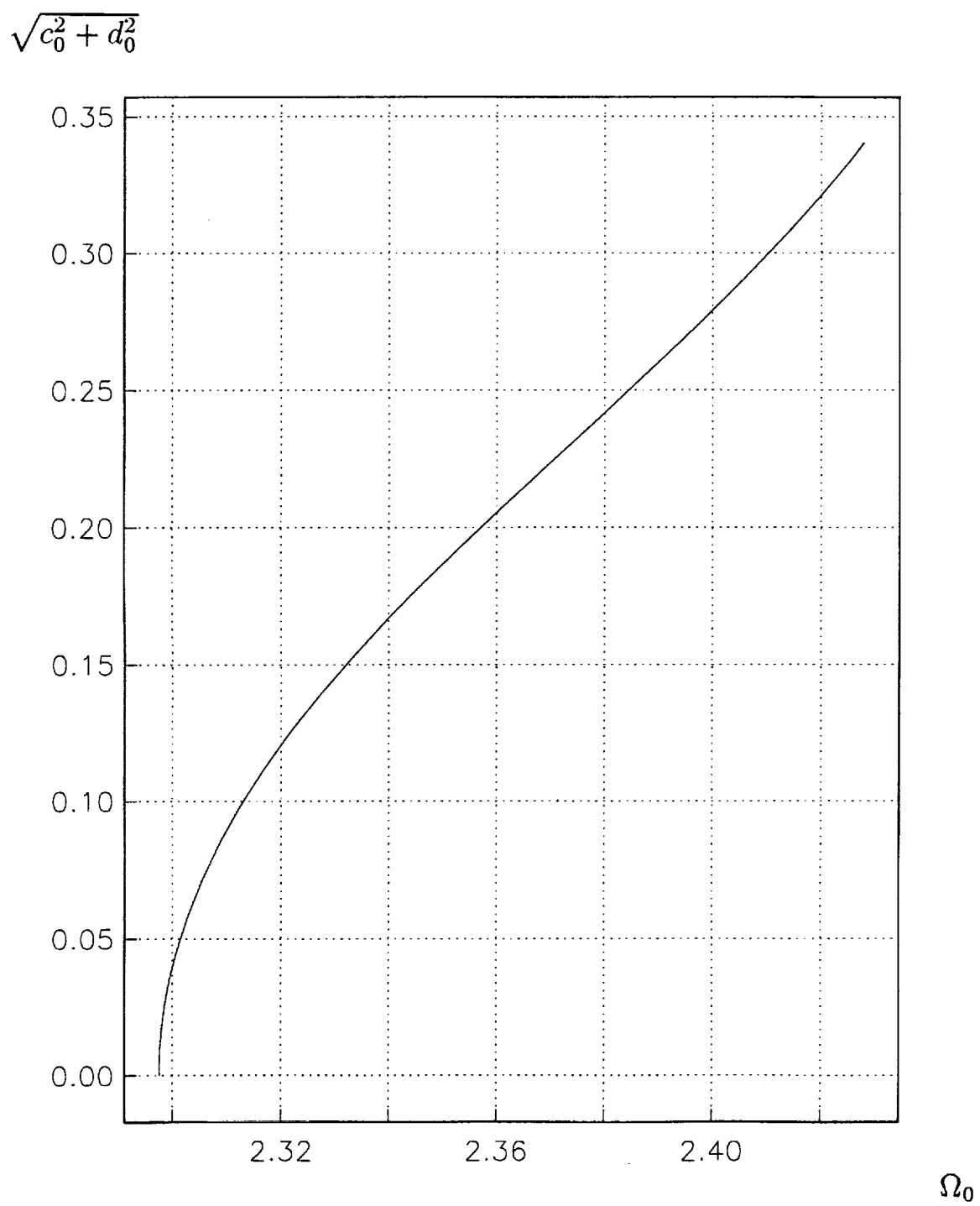


Figure 2 The dependence of  $\Omega_0$  on  $\sqrt{c_0^2 + d_0^2}$ .

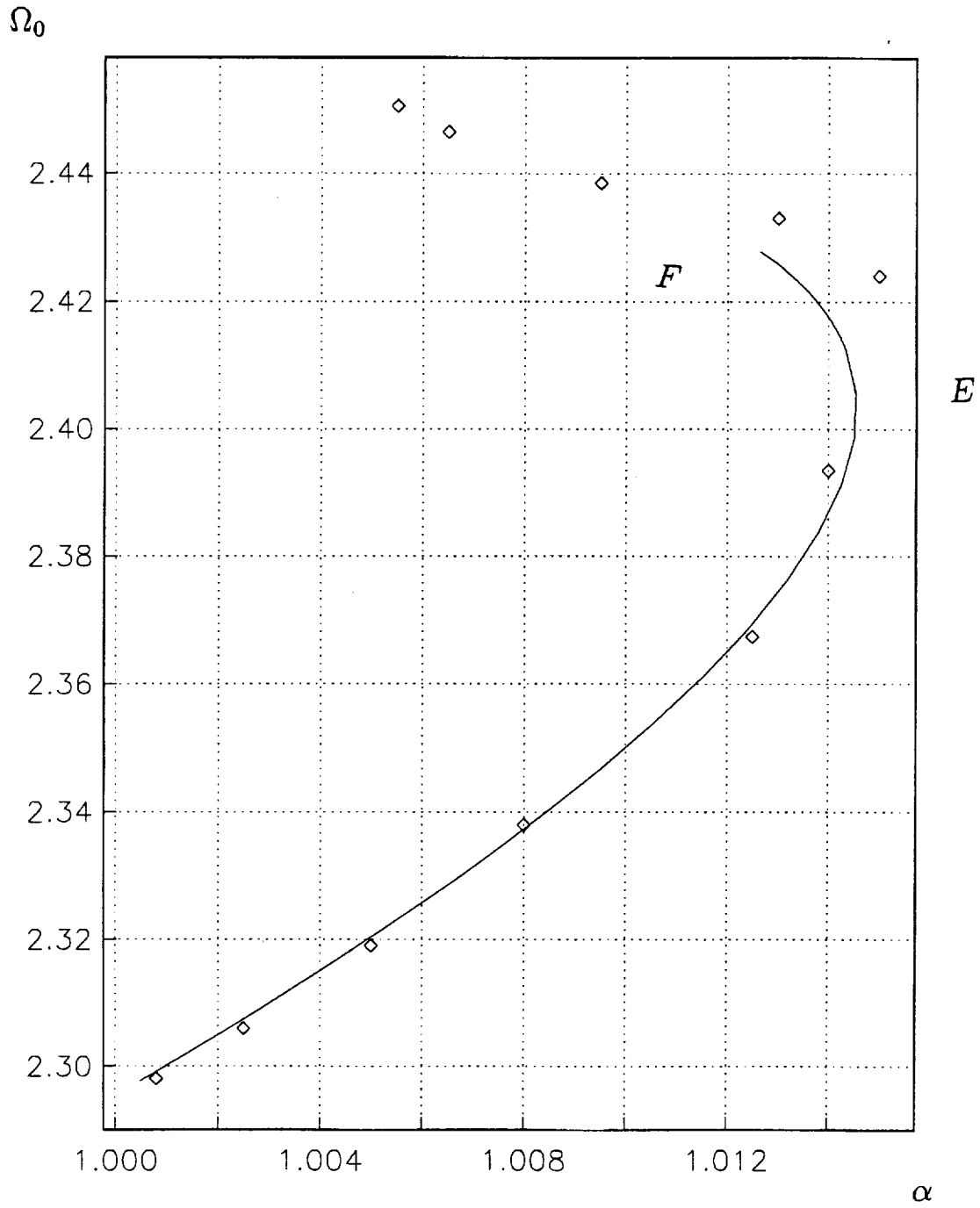


Figure 3 The dependence of  $\Omega_0$  on  $\alpha$ . The symbols denote the results of Conlisk et al.

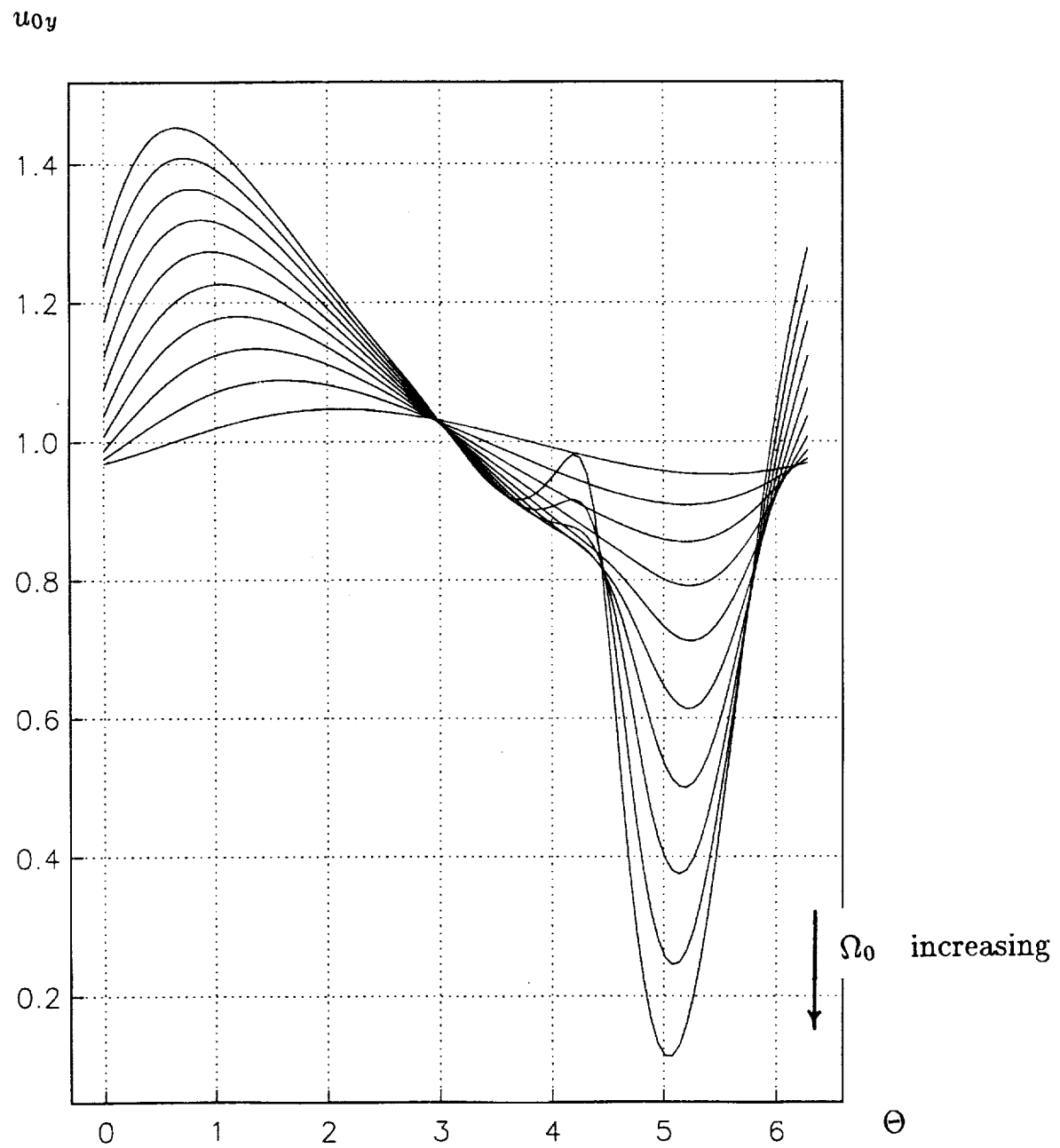


Figure 4 The shear stress as a function of  $\Theta$  for  $\Omega_0 = 2.2995, 2.3041, 2.3125, 2.3245, 2.3398,$

$2.3575, 2.3763, 2.395, 2.4124, 2.4275$

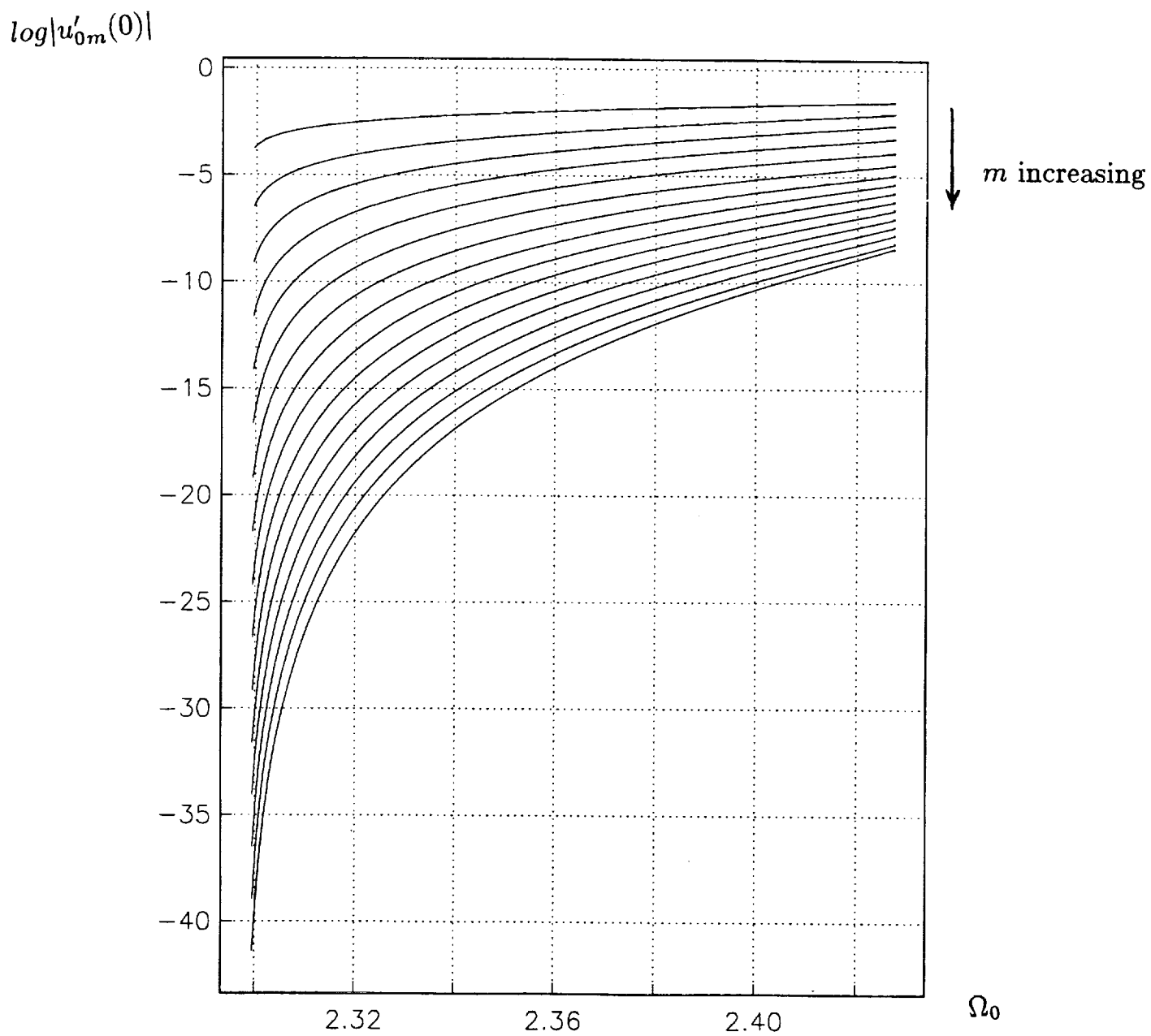


Figure 5 The shear stress  $u'_{0m}(0)$  as a function of  $\Omega_0$  for the first sixteen modes.

$$\frac{\Gamma^2(\frac{2}{3})K}{3^{\frac{1}{3}}}$$

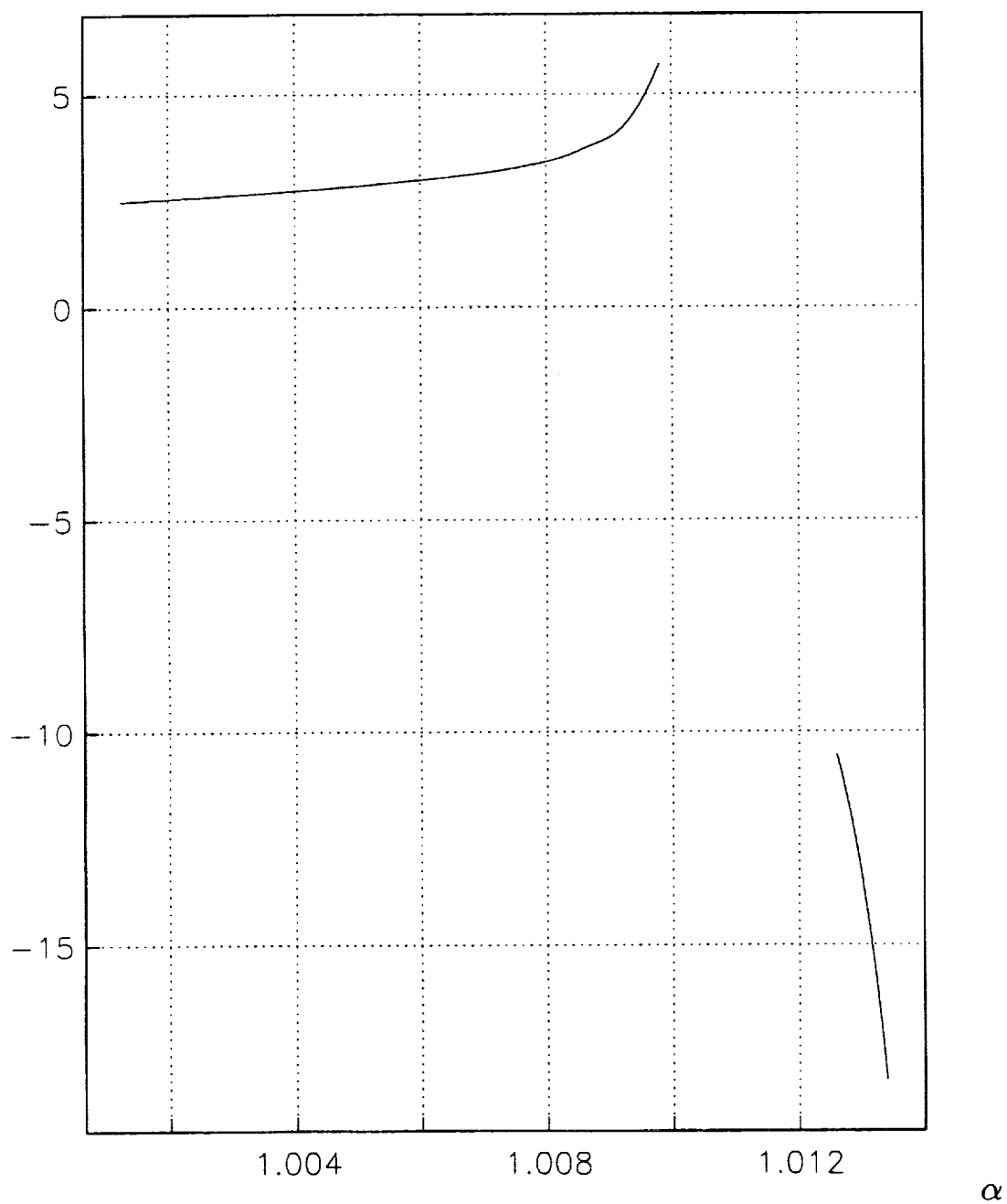


Figure 6 The dependence of  $K$  on  $\alpha$ .



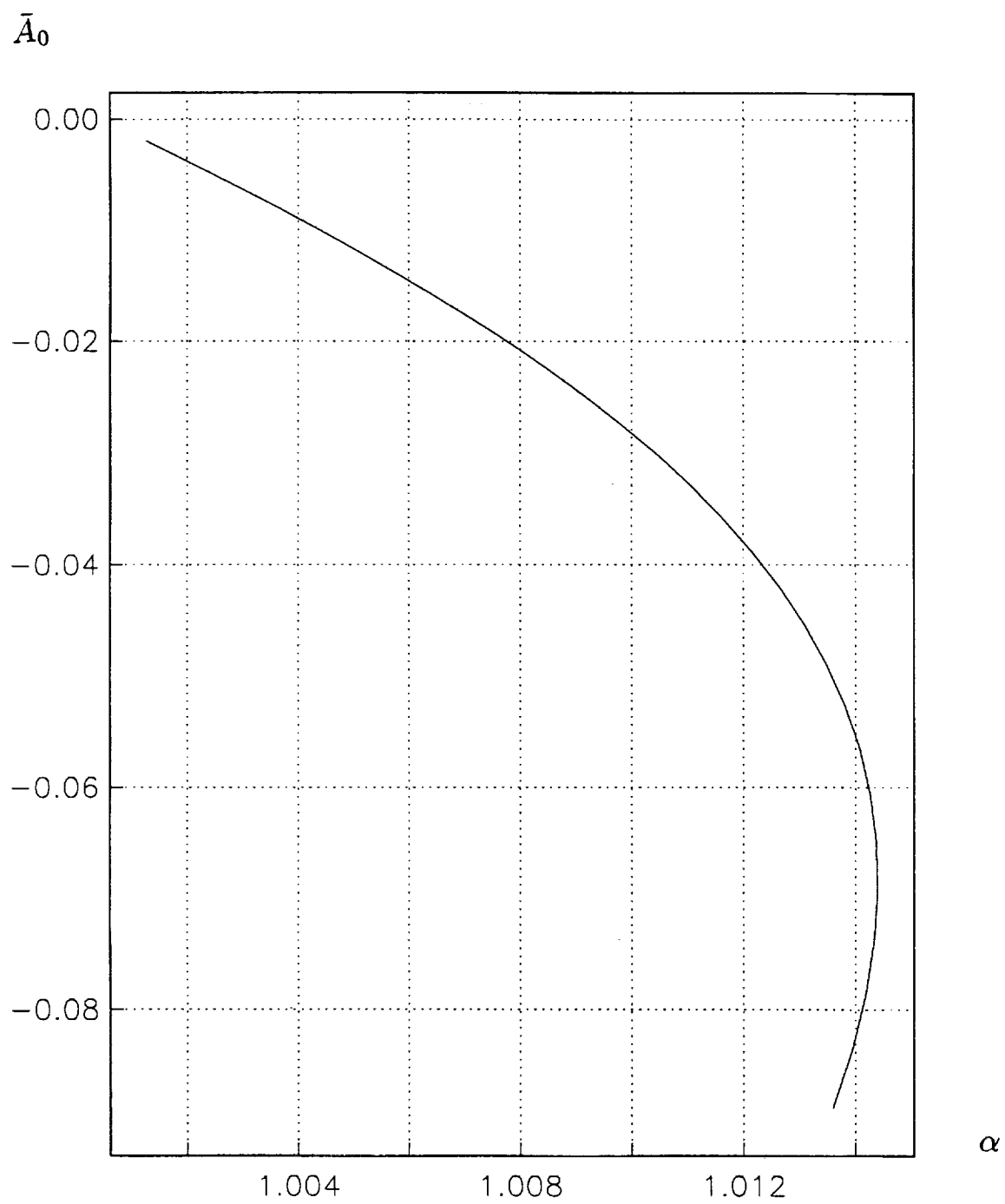


Figure 7 The dependence of  $\bar{A}_0$  on  $\alpha$ .

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13. ABSTRACT (Maximum 200 words) Our concern is with the evolution of large amplitude Tollmien-Schlichting waves in boundary layer flows. In fact the disturbances we consider are of a comparable size to the unperturbed state. We shall describe two-dimensional disturbances which are locally periodic in time and space. This is achieved using a phase equation approach of the type discussed by Howard and Kopell (1977) in the context of reaction-diffusion equations. We shall consider both large and $O(1)$ Reynolds numbers flows though, in order to keep our asymptotics respectable, our finite Reynolds number calculation will be carried out for the asymptotic suction flow. Our large Reynolds number analysis, though carried out for Blasius flow, is valid for any steady two-dimensional boundary layer. In both cases the phase equation approach shows that the wavenumber and frequency will develop shocks or other discontinuities as the disturbance evolves. As a special case we consider the evolution of constant frequency/wavenumber disturbances and show that their modulational instability is controlled by Burgers equation at finite Reynolds number and by a new integro-differential evolution equation at large Reynolds numbers. For the large Reynolds number case the evolution equation points to the development of a spatially localized singularity at a finite time. The three-dimensional generalizations of the evolution equations is also given for the case of weak spanwise modulations.				
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